Grade 11/12 Math Circles
November 23, 2022
Generating Functions - Problem Set

1. Create combinatorial classes (including a set and weight function) to represent the following:

    (a) Rolling a regular 8-sided die.

    (b) Choosing some number of 5 cent coins from an infinitely large pile.

    (c) Drawing a card from a deck with four of each numbered card 1-10.

2. Find the generating functions for the following combinatorial classes:

    (a) 0-1 strings of length two with a weight function of the number of 1's in the string.

    (b) One roll of 2 regular 4-sided dice, where the weight function is the sum of the values rolled.

    (c) 0-1 strings of length 4 with a weight function of the number of occurrences of 01.

3. For $F(z) = 1 + 3z + 6z^2 + 10z^3 + 15z^4 + 21z^5 + 28z^6 + \ldots$, determine:

    (a) $[z^0]F(z)$

    (b) $[z^3]F(z)$

    (c) $[z^9]F(z)$

    (d) $[z^n]F(z)$

4. Write the following generating functions as simply as possible:

    (a) $F(z) = 2z^2 + 4z^3 + 8z^4 + 16z^5 + 32z^6 + \ldots$

    (b) $G(z) = 1 + z^2 + z^4 + z^6 + z^8 + \ldots$

    (c) $H(z) = z + 3z^3 + 9z^5 + 27z^7 + 81z^9 + \ldots$

5. Find a generating function for 0-1 strings beginning with 11 and ending with 0, where the weight function is the length of the string.

6. Consider the combinatorial class of 0-1 strings beginning with 101 and ending with 10, with a weight function corresponding to the length of the string.

    (a) Using a method similar to *Example: Multiplication 2* from the lesson, find a generating function for this combinatorial class.

    (b) How many 0-1 strings of length four start with 101 and end in 10? Is the actual answer the same as in the generating function from part a?

    (c) **Challenge:** If they are different, why might that be? How could it be fixed?

7. Find an expression for the number of ways to make \$2.65 in change (with 5, 10 and 25 cent coins available, as well as 1 and 2 dollar coins).

8. Find 2 four-sided dice such that:

   - Each side has a positive integer number of dots

   - The two dice are not the same

   - The probability of rolling a sum of $2, \ldots, 8$ on these dice is the same as the probabilities for regular four-sided dice

   Hint: $(z + z^2 + z^3 + z^4)^2 = (z^2 + 1)^2(z + 1)^2 z^2$

9. **Challenge:** Find a generating function for the following combinatorial class:

   Set: Ordered lists of length three, where each entry is a positive integer.

   Weight Function: Sum of the integers in the given ordered list.

   Example: (2,3,1) has a weight of 2 + 3 + 1 = 6.